Chance constrained optimization: Theory, Methods, and Applications

(Workshop Tutorial, Oct. 5-7, 2022, TU Ilmenau)

Dr. rer. nat. habil. Abebe Geletu

E-mail: abebe.geletu@tu-ilmenau.de

German Research Chair

AIMS Rwanda

Topics

This Chancy, Chancy, Chancy World?

Leonard Rastrigin, Mir Publishers, 1984.

Uncertainty is the only certainty ...

John Allen Paulos, Temple University, Philadelphia

1. Introduction

What is uncertainty?

A phenomena that cannot be predicted exactly is uncertain.

Sources of uncertainty (some examples)

- measurement errors
- random external disturbances, e.g. wind speed, solar radiation intensity, ambient temperature and pressure, news, market prices, etc.
- model inaccuracy

  A model is only a simplified representation of a system.

- system's inherent uncertainty

  Heisenberg's uncertainty principle: We cannot measure the position and the momentum of a particle with absolute precision

1. Introduction ...Characterization of uncertainties

- Identification and classification of **high-impact uncertainties** is an important initial step.

A rough classification of uncertainties:
measurable and **non-measurable (non probabilistic) uncertainties**

Measurable uncertainties:

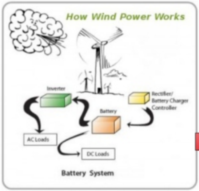
- Uncertainties that can be characterized through statistical measures like: mean, variance, covariance, probability distribution, etc.
  $\Rightarrow$ these are called *random variables*

Non-measurable uncertainties:

- Uncertainties with no definite distributional characteristics
  $\Rightarrow$ uncertainties with no sufficient historical or measurement data; with *ambiguous distributions*, etc.
  $\Rightarrow$ **Best practice**: define a confidence-interval or set to which such uncertainties belong to.

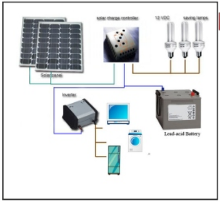
1. Introduction ... Uncertainties in Applications

ξ - wind speed



Power Systems Management
and
Optimal Power Flow

ξ - solar intensity



System description

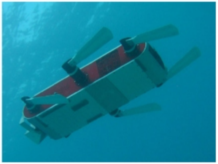
$$\dot{x} = f(x, u, \xi)$$
$$0 = g(x, u, \xi)$$

1. Introduction ... Uncertainties in Applications...

UGV

Optimal Navigation
and
Reliable Obstacle Avoidance

ξ - ramdom obstacles
- sensor measurement errors
- state generated errors



AUV

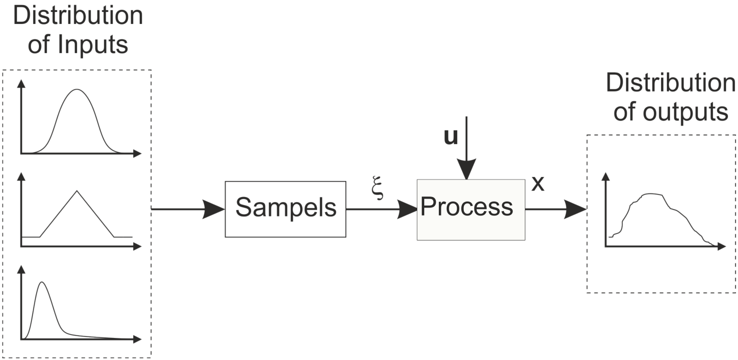
UAV

Systems Desciprion

$$\dot{x} = f(x, u, \xi)$$

$$0 = g(x, u, \xi)$$

1. Introduction - Impact of uncertainties



Consequences of uncertainties:

- future system behaviors cannot be predict precisely
- system becomes unreliable with performance degradation
- risk of violation of constraints
- etc.

2. Optimization under uncertainty - History

Optimization under uncertainty

- Dantzig 1955 (stochastic optimization with recourse)

Chance constrained optimization

- Charnes, Copper & Symonds 1958, 1959.
- Miller and Wagner 1965 (joint chance constraints)

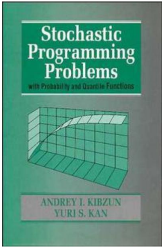
Major contributors to CCOPT since the 1970's:

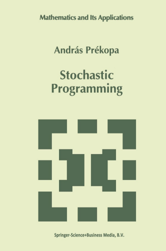
- Prekopa and associates 1972, 1973, 1995, 2001, 2011.
- Raik 1971, 1975
- Kall and Wallace
- Wets
- Henrion, Römisch, and associates
- Nemirovski, Shapiro, and associates

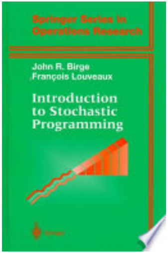
Robust Optimization

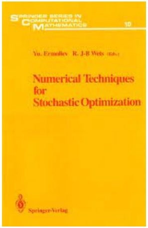
- Ben-Tal, Nemirovski, Bertsimas,...

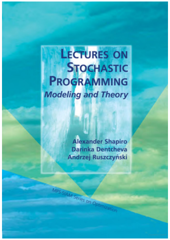
2. Optimization under uncertainty ... Text books and references

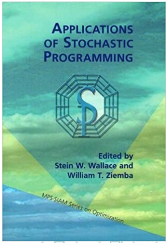




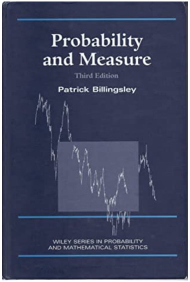


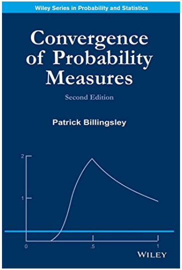


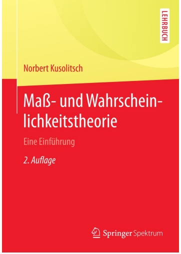


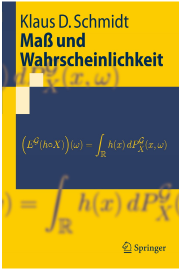


2. .... Text books and references...









2. .... Helpful Prerequisites

- Mathematical analysis (real and functional)
- Mathematical optimization - theory and methods
- Convex analysis
- Set-valued analysis
- Optimization on function spaces (e.g., PDE constrained optimization)
- Predictive control of lumped (DAEs) and distributed parameter (PDEs) systems

2. Optimization under uncertainty...

- **Performance criteria**: $f(x, u, \xi)$
- **Process model**: $G(x, u, \xi) = 0$
- **Constraints**: $g(x, u, \xi) \leq 0$

$u \in U.$

Question

What is the best mathematical model for optimization under uncertainty?

Answer

- No conclusive answer!
- It depends on type of application, available information on uncertainties, purpose of optimization, etc.

2. Optimization under uncertainty...

Optimization with the **nominal values** $\mu = E[\xi]$ for the uncertainties.

Standard deterministic

$$\min_{u \in U} f\left(x, u, \mu\right)$$

subject to:

$$G(x, u, \mu) = 0$$

$$g(x, u, \mu) \leq 0.$$

★

- This is a "plain vanilla" optimization problem.
- It does not seriously account for the impact of uncertainties and system robustness.

2. Optimization under uncertainty...

Meaningless

$$\min_{u \in U} \; f(x, u, \xi)$$

subject to:

$$G(x, u, \xi) = 0$$
$$g(x, u, \xi) \leq 0$$

Worst-case

$$\min_{u \in U} \max_{\xi \in \Omega} f(x, u, \xi)$$

subject to:

$$G(x, u, \xi) = 0$$
$$\max_{\xi \in \Omega} g(x, u, \xi) \leq 0$$

50% Reliability

$$\min_{u \in U} E\left[f(x, u, \xi)\right]$$

subject to:

$$E\left[G(x, u, \xi)\right] = 0$$
$$E\left[g(x, u, \xi)\right] \leq 0$$

Robust Optimization/Semi-infinite Optim.

$$\min_{u \in U} E\left[f(x, u, \xi)\right]$$

subject to:

$$G(x, u, \xi) = 0$$
$$g(x, u, \xi) \leq 0, \forall \xi \in \Omega.$$

2. Optimization under uncertainty...

Robust Optimization

Robustness

As the going gets tough, the tough gets going.
Billy Ocean, 1985.

Hold on what ever happens:

$$g(x,u,\xi) \leq 0, \forall \xi \in \Omega \iff \max_{\xi} g(x,u,\xi) \leq 0.$$

$\Rightarrow$ **Take no risk of constraint violation**. **Expensive!!!**

But, practically and frequently, under uncertainty

- constraints on future outcomes are bound to be violated
- instead opt for reliability and fault tolerance
- In fact, **No Risk, No Fun!**

Chance constrained optimization - General form

$$(CCOPT) \qquad \min_u E\left[f\left(x,u,\xi\right)\right] \tag{1}$$

subject to:

$$G(x,u,\xi)=0, \tag{2}$$

$$Pr\left\{g(x,u,\xi)\leq 0\right\}\geq \alpha, \tag{3}$$

$$u\in U. \tag{4}$$

The major objects of investigations in CCOPT are chance constraints.

3. Chance constrained optimization...

Basic Assumptions

- $u \in \mathbb{R}^m$ is a vector of deterministic decision variables
- $x \in \mathbb{R}^n$ is a vector of random state variables (due to eqn. (54))
- U is a compact subset of $\mathbb{R}^m$,
- $(\Omega, \mathcal{A}, Pr)$ - a complete **probability space**,
- $\mathcal{A}$ is a σ-algebra
- $\Omega \subset \mathbb{R}^p$ is a Borel set,
- $Pr : \mathcal{A} \to [0, 1]$ **probability measure**,
- ξ - a random variable with a **continuous joint pdf** $\phi(\xi)$,
- $Pr\{A\} = \int_A \phi(\xi) d\xi$, for $A \in \mathcal{A}$,
- $E[\cdot]$ - the **expected-value** operator,
- $f, G, g : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ are at least **one-time differentiable**

3. Chance constrained optimization ...

The expression $Pr\{g(x,u,\xi) \le 0\}$ represents

$$Pr\{\xi \in \Omega \mid g(x,u,\xi) \le 0\}$$

- **Chance (or probabilistic) constraint**

$$Pr\{g(x,u,\xi) \le 0\} \ge \alpha,$$

  where $\alpha \in [0,1]$ is a pre-given (user-defined) **probability (reliability) level**
- The probability (reliability) level α is usually $\frac{1}{2} \le \alpha \le 1$; i.e. **above average reliability**. Commonly, $\alpha = 0.95$, $\alpha = 0.98$, $\alpha = 0.99$, etc.
- α near 1 is too conservative, $\alpha = 1$ deterministic.

3. Chance constrained optimization ...Definitions

- **Probability function**

$$p(u) := Pr\left\{g(x, u, \xi) \leq 0\right\}$$

of the chance constraint in CCOPT.

- **Feasible set** of CCOPT is

$$\mathcal{P} := \left\{u \ \in U | \ p(u) \geq \alpha\right\}.$$

Standard assumptions

- The probability measure $Pr(\cdot)$ is associated with the joint pdf $\phi(\cdot)$ through

$$dPr(\xi) = \phi(\xi)d\xi$$

  (**Radon−Nikodym Theorem**)
- **Measure-zero property**
  Given u and x

$$Pr\{\xi \in \Omega \mid g(x, u, \xi) = 0\} = 0$$

3. CCOPT ... Equivalent representations

- **(probability of to be) + (probability of not to be) = 1**

$$Pr\left\{g(u,x,\xi)\leq 0\right\}+Pr\left\{g(u,x,\xi)\geq 0\right\}=1$$

Hence, $Pr\left\{g(u,x,\xi)\leq 0\right\}\geq\alpha$ is **equivalent to**

$$Pr\left\{g(u,x,\xi)\geq 0\right\}\leq 1-\alpha.$$

- Integral representation of chance constraints

$$
\begin{aligned}
Pr\{g(u,x,\xi) \le 0\} &= Pr\{\xi \in \Omega \mid g(u,x,\xi) \le 0\} \\
&= \int_{\{\xi \in \Omega \mid g(u,x,\xi) \le 0\}} \phi(\xi) d\xi \ge \alpha.
\end{aligned}
$$

where $\phi(\xi)$ is the probability density function of ξ.

3. CCOPT ... Reliability and Risk

- **No reliability**: If $\alpha = 0$, then $\mathcal{P} = U \subset \mathbb{R}^m$

$\Rightarrow$ **With no guarantee, any decision is feasible!**

$\Rightarrow$ **100 % Risky!**

- **No Risk**: If $\alpha = 1$, then

$$Pr\{g(u,x,\xi) \leq 0\} \geq 1 \quad \Longrightarrow \quad Pr\{g(u,x,\xi) \leq 0\} = 1.$$

$\Rightarrow$ For $\alpha = 1$, CCOPT is a **robust optimization** problem!

$\Rightarrow$ That is, $\alpha = 1$ in CCOPT leads to a ***conservative decision***.

Let $\mathbb{I}(s) := \begin{cases} 1, & \text{if } s \leq 0, \\ 0, & \text{if } s > 0 \end{cases}$. Then
$E[I(g(x,u,\xi))] = Pr\{g(x,u,\xi) \leq 0\}$.

$$\int_\Omega I(g(x,u,\xi))\phi(\xi)d\xi = 1 = \int_\Omega \phi(\xi)d\xi$$
$$\Rightarrow \int_\Omega [I(g(x,u,\xi)) - 1]\,\phi(\xi)d\xi = 0$$
$$\Rightarrow [I(g(x,u,\xi)) - 1] = 0, \text{ for almost all} \xi \in \Omega$$
$$\Rightarrow g(x,u,\xi) \leq 0, \text{ for almost all } \xi \in \Omega$$

3. CCOPT ... Single or Joint Chance Constraints

In case of several random inequality constraints:
use either

(i) **Single chance constraints**

$$Pr\{g_i(x,u,\xi) \leq 0\} \geq \alpha_i, i = 1, \ldots, m.$$

or

(ii) **Joint chance constraints**

$$Pr\{g_i(x,u,\xi) \leq 0, i = 1, \ldots, m\} \geq \alpha.$$

Joint-chance constraints pose more difficulties. We focus here **only on single** chance constrained problems.

3. Chance constrained optimization - compact form

- **Assumption** : Under some standard assumptions (e.g., **IFT**), the equation $G(x, u, \xi) = 0$ is can be solved to obtain $x(u, \xi)$, for a given u and a realization of ξ. So that

  $$f(u, \xi) = f\left(x(u, \xi), u, \xi\right) \text{ and } g(u, \xi) := g(x(u, \xi), u, \xi)$$

The standard form

$$(CCOPT) \qquad \min_u E\left[f(u, \xi)\right] \tag{5}$$

subject to:

$$Pr\left\{g(u, \xi) \leq 0\right\} \geq \alpha \tag{6}$$

$$u \in U. \tag{7}$$

3. Chance constrained optimization ...

Objective

To determine an optimal decision variable u^* for the objective function $E\left[f(u,\xi)\right]$ guarantee the satisfaction of the chance constraint $Pr\left\{g(u,\xi) \leq 0\right\} \geq \alpha$ with a given probability level α.

$\Rightarrow$ The optimal decision variable u^* gurantees $Pr\left\{g(u,\xi) \leq 0\right\} \geq \alpha$; while accepting a **(possible risk of) violation** of constraints by $(1-\alpha)$ i.e., $Pr\left\{g(u^*,\xi) > 0\right\} \leq 1-\alpha$.

In the face of uncertainty, allowing a degree of constraints violation may yield a performance gain, but it may still constitute some consequences (risk)

Chance constrained optimization with variance minimization

A general form

$$(CCOPT) \qquad \min_u \{\gamma_1 E\left[f(u,\xi)\right] + \gamma_2 Var\left[f(u,\xi)\right]\} \quad (8)$$

$$s.t.$$

$$Pr\{g(u,\xi) \leq 0\} \geq \alpha \quad (9)$$

$$u \in U. \quad (10)$$

- The parameters $\gamma_1, \gamma_2 \geq 0$ are weighing factors.
- By choosing a larger value for either γ_1 or γ_2, we can decide which one ($E\left[f(u,\xi)\right]$ or $Var\left[f(u,\xi)\right]$) we would like to minimize the most.
- Traditionally, we have $\gamma_1 = 1$ and $\gamma_2 = 0$ so that the objective consists of only $E\left[f(u,\xi)\right]$ (**standard CCOPT**).
- If $f(u,\xi) = f(u)$, then we have $E[f(u,\xi)] = f(u)$ and $Var[f(u,\xi)] = Var[f(u)] = 0$.

A simple example - (a) Random decision

Consider the problem

$$(P) \qquad \max_{u \in \mathbb{R}^2} \{u_1 + u_2\} \tag{11}$$

subject to:

$$u_1^2 + u_2^2 \leq 5 + \xi. \tag{12}$$

For each fixed value of ξ, the optimal solution (decision) will be

$$u^* = \left(\sqrt{\frac{5+\xi}{2}}, \sqrt{\frac{5+\xi}{2}} \right).$$

If ξ is random, then u^* is a random decision. **Dangerous!**

A simple example ... (b) chance constrained optimization

Suppose $\xi \sim \mathcal{N}(0,1)$ with distribution function $\Phi(x) = \int_{-\infty}^{x} \phi(\xi) d\xi$.

$$(CCOPT) \qquad \max_{u \in \mathbb{R}^2} \{u_1 + u_2\} \tag{13}$$

subject to:

$$Pr\{u_1^2 + u_2^2 \leq 5 + \xi\} \geq \alpha. \tag{14}$$

Note that

$$Pr\{u_1^2 + u_2^2 \leq 5 + \xi\} = 1 - Pr\{u_1^2 + u_2^2 > 5 + \xi\} = 1 - \Phi(u_1^2 + u_2^2 - 5)$$

Hence,

$$Pr\{u_1^2 + u_2^2 \leq 5 + \xi\} \geq \alpha \quad \equiv \quad u_1^2 + u_2^2 - 5 \leq \Phi^{-1}(1-\alpha).$$

Hence, optimal solution of CCOPT

$$u_{cc}^*(\alpha) = \left(\sqrt{\frac{5+\Phi^{-1}(1-\alpha)}{2}}, \sqrt{\frac{5+\Phi^{-1}(1-\alpha)}{2}} \right)$$

and optimal objective function value $f(u_{cc}^*(\alpha)) = 2\sqrt{\frac{5+\Phi^{-1}(1-\alpha)}{2}}$.

A simple example ... (c) Robust optimization formulation

$$(RO) \qquad \max_{u\in\mathbb{R}^2} \{u_1 + u_2\} \tag{15}$$

subject to:

$$u_1^2 + u_2^2 \le 5 + \xi, \forall \xi \in \Omega, \tag{16}$$

for $\Omega = [\mu_\xi - 4\sigma_\xi, \mu_\xi + 4\sigma_\xi]$ - a 99% confidence interval for $\xi \sim \mathcal{N}(0,1)$.

Feasible set of RO:

$$\bigcap_{\xi\in[-4,4]} \left\{u \in \mathbb{R}^2 \mid u_1^2 + u_2^2 \le 5 + \xi\right\} = \left\{u \in \mathbb{R}^2 \mid u_1^2 + u_2^2 \le 1\right\}.$$

Hence, solution $u_{RO}^* = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ with the optimal objective function value $f(u_{RO}^*) = \sqrt{2}$.

A simple example ...(d) Comparison of CC and RO

- For $\alpha = 0.98$, the optimal value of CC

$$f(u^*_{cc}(0.98)) = 2\sqrt{\frac{5 + \Phi^{-1}(0.02)}{2}} \approx 2\sqrt{\frac{5 - 2.0537}{2}} \approx 2.4275$$

while $f(u^*_{RO}) = \sqrt{2}$.

- accepting a risk of constraint-violation by $1 - \alpha = 0.02$ brings a profit $f(u^*_{cc}(0.98)) = 2.4275$.
- while taking no risk brings the lower profit $f(u^*_{RO}) = \sqrt{2}$.

$\Longrightarrow$ In scale of millions, the difference can be enormous.

Caution::
Not all CCOPT problems are simple to solve like (13)-(14).

Problems that lead to CCOPT ... Probability maximization

Maximum probability problem

- Find a decision u to for a high probability

$$\max_{u \in \mathbb{R}^m} Pr\{g(u,\xi) \leq 0\}$$

of winning a game, a lottery, attaining a goal, etc.
Can be written as

$$(CCOPT) \qquad \max_{u \in \mathbb{R}^m, \alpha \in [0,1]} \alpha \tag{17}$$

subject to:

$$Pr\{g(u,\xi) \leq 0\} \geq \alpha. \tag{18}$$

Problems that lead to CCOPT ...Portfolio optimization

Portfolio optimization

Total value of a portfolio at the end of a given time period:

$$f(u, \xi) = \sum_{i=1}^{n} u_i \xi_i = \xi^\top u$$

- Objective function: $E\left[\xi^\top u\right] \rightarrow$ maximize.
- $u_i, i = 1, \dots, n$, investment proportions on n assets
- K desired total return
- Chance constraint:

$$Pr\{f(u, \xi) < K\} \leq 1 - \alpha \Leftrightarrow Pr\{f(u, \xi) \geq K\} \geq \alpha$$

Optimization problem:

$$(CCOPT) \qquad \max_{u \in U} E\left[\xi^\top u\right] \tag{19}$$

$$s.t. \quad \mathbf{e}^\top u = 1 \tag{20}$$

$$Pr\{\xi^\top u \geq K\} \geq \alpha. \tag{21}$$

Problems that lead to CCOPT ...Financial Risk Metrics

Value at Risk (VaR)

<u>Mathematical definition</u>

- Let $g : U \times \Omega \to \mathbb{R}$ - scalar valued function
- $g(u, \xi)$ defines **a loss under a strategy** $u \in U$.

The expression

$$Pr\{g(u, \xi) \leq \gamma\}$$

represents the probability of the loss $g(u, \xi)$ to lie below an admissible loss level γ.

VaR

$$VaR(g(u, \xi); \alpha) = \inf_{\gamma} \{\gamma \mid Pr\{g(u, \xi) \leq \gamma\} \geq \alpha\}.$$

$$\equiv \sup_{\gamma} Pr\{g(u, \xi) > \gamma\} \leq 1 - \alpha.$$

$\Rightarrow$ **The worst loss $g(u, \xi)$, for a decision u, is expected to occur with a probability level less than $1 - \alpha$.**

Problems that lead to CCOPT ... Risk Metrics

Conditional value-at-risk (cVaR)

Conditional value-at-risk of a portfolio is the expected return, conditioned on the return being less than or equal to VaR.

Pflug 2000 (Optimization Formulation)

The **conditional value at risk** (cVaR) of a random variable $Z = g(u, \xi)$ is defined as

$$cVaR(Z;\alpha) = \inf_{\beta} \left\{ \beta + \frac{1}{1-\alpha} E\left[(Z-\beta)_+\right] \right\}$$

In general we have $cVaR(Z;\alpha) \geq VaR(Z;\alpha)$.

References:

- G.C. Pflug: Some Remarks on the Value-at-Risk and on the Conditional Value-at-Risk. In: Probabilistic Constrained Optimization: Methodology and Applications, (Uryasev ed), Kluwer, 2000
- R.T. Rockafellar and S. Uryasev: Optimization of Conditional Value-At-Risk. Journal of Risk 2 (4), 21-51,

Chance Constrained Optimization ...Properties..

Fundamental questions

Given $p(u) = Pr\{g(u,\xi) \leq 0\}$.

- Does $p(u)$ have an analytic expression?
- When is $p(u)$ continuous?
- What are the convexity properties of $p(u)$ and when is CCOPT a convex optimization problem?
- When is $p(u)$ differentiable?
- How to evaluate $p(u)$ for a given u?
- How to solve the optimization problem CCOPT?

4. Linear Chance Constrained Optimization

(A) Linear CCOPT with single chance constraints

$$(LCCOPT) \qquad \min_{u \in \mathbb{R}^m} E\left[c^\top u\right] \tag{22}$$

$$s.t.$$

$$Pr\{a_i^\top u \leq b_i\} \geq \alpha_i, i = 1, \ldots, m \tag{23}$$

$$u \geq 0, \tag{24}$$

where:

- the decision variable $u = (u_1, \ldots, u_n)^\top \in \mathbb{R}^n$ is deterministic;
- any one or a combination of the matrix $A = \begin{bmatrix} a_1^\top \\ a_2^\top \\ \vdots \\ a_m^\top \end{bmatrix}$, the vectors $c \in \mathbb{R}^n$ or $b \in \mathbb{R}^m$ can be random.

We consider three cases.

(i) The matrix $A = (a_{ij})$ is random, b and c are deterministic.
Assumption: The elements a_{ij} of the matrix A are independently normally distributed with mean μ_{ij} and standard deviation σ_{ij}; i.e. $a_{ij} \backsim \mathcal{N}(\mu_{ij}, \sigma_{ij})$.

- Define $d_i := a_i^\top u = \sum_{j=1}^n u_j a_{ij}, i = 1, \ldots, n$.
- d_i is a linear combination of normally distributed random variables.
- For each $i \in \{1, 2, \ldots, n\}$, d_i is normally distributed; with
  - mean: $\mu_{d_i} = \sum_{j=1}^n \mu_{ij} x_j$ (**Exercise!**)
  - variance (standard deviation): $\sigma_{d_i}^2 = \sum_{j=1}^n \sigma_{ij} x_j^2$(**Exercise!**)

Hence,

$$Pr\left\{a_i^\top u \le b_i\right\} \ge \alpha_i \equiv Pr\left\{\sum_{j=1}^n a_{ij} u_j \le b_i\right\} \ge \alpha_i \equiv Pr\left\{d_i \ \le b_i\right\} \ge \alpha_i,$$

LCCOPT ... Deterministic representation...

$$\equiv Pr\left\{\frac{d_i-\mu_{d_i}}{\sigma_{d_i}} \leq \frac{b_i-\mu_{d_i}}{\sigma_{d_i}}\right\} \geq \alpha_i, i=1:m.$$

- The random variable $z_i := \left(\frac{d_i-\mu_{d_i}}{\sigma_{d_i}}\right)$ has a standard normal distribution, i.e. $z_i \backsim \mathcal{N}(0;1), i=1:m$.

  $\Rightarrow$

  $$Pr\left\{\frac{d_i-\mu_{d_i}}{\sigma_{d_i}} \leq \frac{b_i-\mu_{d_i}}{\sigma_{d_i}}\right\} = \Phi\left(\frac{b_i-\mu_{d_i}}{\sigma_{d_i}}\right),$$

  where $\Phi(\cdot)$ is the **cumulative standard normal distribution function** of z_i.

  - ▶ Consequently, we have

  $$\Phi\left(\frac{b_i-\mu_{d_i}}{\sigma_{d_i}}\right) \geq \alpha_i, i=1,\ldots,m.$$

  $\Leftrightarrow$

  $$\frac{b_i-\mu_{d_i}}{\sigma_{d_i}} \geq \Phi^{-1}(\alpha_i) \Leftrightarrow b_i-\mu_{d_i} \geq \sigma_{d_i}\Phi^{-1}(\alpha_i), i=1,\ldots,m.$$

LCCOPT ... Deterministic representation...

where Φ^{-1} is the **inverse standard normal distribution** function.

$$b_i - \mu_{d_i} \geq \sigma_{d_i}\Phi^{-1}(\alpha_i)$$
$$\Leftrightarrow \sum_{j=1}^{m} \mu_j u_j + \Phi^{-1}(\alpha_j)\sqrt{\sum_{j=1}^{m} \sigma_{ij}^2 u_j^2} - b_i \leq 0.$$

- Therefore, an equivalent representation of (LCCOPT) is

$$(NLP) \qquad \min_{x\in\mathbb{R}^n} c^\top u$$
$$s.t.$$
$$\sum_{j=1}^{m} \mu_j u_j + \Phi^{-1}(\alpha_j)\sqrt{\sum_{j=1}^{m} \sigma_{ij}^2 u_j^2} - b_i \leq 0, i = 1, 2, \ldots, m,$$
$$u \geq 0.$$

This a **deterministic nonlinear optimization problem** (it is convex if $\alpha \in [0.5, 1]$ **not convex for** $0 < \alpha < 0.5$).

LCCOPT ... Deterministic representation...

(ii) The vector b is random, the matrix $A = (a_{ij})$ is and the vector c are deterministic.
Assumption: The components of the vector $b^\top = (b_1, b_2, \ldots, b_m)$ are independently normally distributed with mean μ_i and variance $\sigma_i, i = 1, 2, \ldots, m$.

- Hence,

$$Pr\left\{a_i^\top u \leq b_i\right\} = Pr\left\{\sum_{j=1}^{n} a_{ij}^\top u_j \leq b_i\right\}$$

$$= Pr\left\{\frac{\sum_{j=1}^{n} a_{ij}^\top u_j - \mu_{b_i}}{\sigma_{b_i}} \leq \frac{b_i - \mu_{b_i}}{\sigma_{b_i}}\right\} \geq \alpha_i$$

$$\Leftrightarrow Pr\left\{\frac{b_i - \mu_{b_i}}{\sigma_{b_i}} \leq \frac{\sum_{j=1}^{n} a_{ij}^\top u_j - \mu_{b_i}}{\sigma_{b_i}}\right\} \leq 1 - \alpha_i .$$

LCCOPT ... Deterministic representation...

Since each $\frac{b_i - \mu_{b_i}}{\sigma_{b_i}}$ have a standard normal distributor, it follows

$$\Phi\left(\frac{\sum_{j=1}^{n} a_{ij}^{\top} u_j - \mu_{b_i}}{\sigma_{b_i}}\right) \leq 1 - \alpha_i \Leftrightarrow \sum_{j=1}^{n} a_{ij}^{\top} u_j - \mu_j \leq \Phi^{-1}(1-\alpha_i)\sigma_{b_i}.$$

- As a result we obtain

$$(LP) \qquad \min_{x \in \mathbb{R}^n} c^{\top} u \tag{25}$$

$$s.t.$$

$$a_i^{\top} u \leq \mu_{b_i} + \Phi^{-1}(1-\alpha_i)\sigma_{b_i}, i = 1, 2, \ldots, m. \tag{26}$$

$$u \geq 0, \tag{27}$$

which is a **deterministic linear optimization problem**.

LCCOPT ... Deterministic representation...

(iii) $\xi = (a_1, a_2, \ldots, a_n, b)$ random and
$g(x, \xi) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n - b$ (a linear function).

Theorem

If the components of the vector ξ are independently normally distributed random variables, then

$$Pr\{g(x,\xi)\} \geq \alpha$$
$$\sum_{i=1}^{n} E[a_i]\, x_i + \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^{n} Var[a_i]\, x_i^2 + Var[b]} \leq E[b].$$

Proof.

Exercise! □

Note: For $\alpha \in [0, 1]$, the values of $\Phi^{-1}(\alpha)$ can be read from look-up tables.
Also use the Matlab function: `y = norminv(p,mu,sigma)`.

LCCOPT ... Deterministic representation...

(iv) The cost vector $a^\top = (a_1, a_2, \dots, a_n) =: \xi^\top$ is random.

Theorem

Suppose $\xi^\top = a^\top = (a_1, a_2, \dots, a_n)$ be normally distributed (not necessarily independent) random variables with mean μ and covariance matrix Σ. Then the feasible set

$$\mathcal{P} = \{u \in \mathbb{R}^m \mid p(u) \geq \alpha\},$$

where $p(u) = Pr\{a^\top u \leq b\} \geq \alpha$ and $b \in \mathbb{R}$, can be exactly represented by

$$\mathcal{P} = \left\{u \in \mathbb{R}^m \mid \mu^\top u + \Phi^{-1}(\alpha)\sqrt{u^\top \Sigma u} \leq b\right\}$$

Proof. (**Exercise!**)

(i) First show that:

$$a \in \mathbb{R}^n, a \sim \mathcal{N}(\mu^\top, \Sigma) \Rightarrow a^\top u - b \in \mathbb{R} \text{ and } a^\top u - b \sim \mathcal{N}(\mu^\top u - b, u^\top \Sigma u)$$

(ii) Consider the two cases: when $u^\top \Sigma u = 0$ and $u^\top \Sigma u \neq 0$, and apply similar techniques as in above.

Exercises:

1. Let

$$Pr\{a_1u_1 + a_2u_2 + a_2u_3 \leq b\} \geq 0.95 \qquad (2)$$

where a_1, a_2, a_3 and b have the distributions $\mathcal{N}(1;1), \mathcal{N}(2;1), \mathcal{N}(3;1)$ and $\mathcal{N}(4;1)$, respectively.
Verify that this chance constraint is equivalent to

$$u_1 + 2u_2 + 3u_3 + 1.645\sqrt{u_1^2 + u_2^2 + u_3^2 + 1} \leq 4.$$

2. Write the deterministic represetnation for chance constraints

$$Pr\{3u_1 + 4u_2 \leq \xi_1\} \geq 0.8$$
$$Pr\{3u_1^2 - u_2^2 \leq \xi_2\} \geq 0.9,$$

where $\xi_1 \backsim \mathcal{N}(0;2)$ and $\xi_2 \backsim \mathcal{N}(0.5;10)$.

LCCOPT ... Deterministic representation...

- Closed-form exact deterministic representation is available mainly when $g(u, \xi)$ a special linear form w.r.t. ξ and $\phi(\xi)$ Gaussian.
- Multiplicative uncertainties, usually lead to nonlinear determinstic represetntions. (cases (ii)-(iv))

- Genrally exact closed form deterministic representation is, generally, not available; especially, in the presence of non-Gaussian ξ.

5. Structural properties of chance constraints

Major references on chance constrained optimization problems

- **Continuity**
  - A. I. Kibzun; Y. S. Kan: Stochastic programming problems John Wiles & Sons, 1996.
  - J. R. Birge; F. Louveaux: Introduction to stochastic programming. Springer-Verlag, 1997.
  - A. Prekopa 1995: Stochastic programming. Kluwer Academic Publishers, 1995.
- **Differentiability** of $p(\cdot)$
  - K. Marti: Differentiation formulas for probability functions. Mathematical Programming, 75(2), 201-220, 1996.
  - S. Uryasev: Derivatives of probability functions and some applications. Annals of Operations Research, 56(1): 287-311, 1995.
- **Convexity** of $p(\cdot)$ and the feasible set $\mathcal{P}$
  - A. Prekopa 1995: Stochastic programming. Kluwer Academic Publishers, 1995.

Continuity

Remark: If $p(x) = Pr\{g(u,\xi) \leq 0\}$ is an upper semi-continuous function, then feasible set $\mathcal{P} = \{u \in U \mid p(u) \geq \alpha\}$. is a closed set.

Theorem (Kall 1987)

*If $g : U \times \Omega \to \mathbb{R}$ is continuous, U and Ω are closed sets, then $p(u)$ is **upper semi-continuous** and the feasible set*

$$\mathcal{P} = \{u \in U \mid p(u) \geq \alpha\}$$

is a closed set.

Proof idea:

- The set-valued map $M(u) = \{\xi \in \Omega \mid g(u,\xi) \leq 0\}$ is upper semi-continuous and closed valued and $M(u)$ is measurable (see, Castaing and Valadier, Geletu).
- Thus for a given a given sequence $u_n \to u_0$,

$$\limsup_{n\to+\infty} M(u_n) = \bigcap_{N\geq 1} cl\left(\bigcup_{n\geq N} M(u_n)\right) \subset M(u_0)$$

- Then show that $\limsup_{n\to\infty} p(u_n) \leq p(u_0)$; i.e. $p(\cdot)$ is u.s.c.

References:

• P. Billingsley: Convergence of probability measures. Wiley & Sons, 1962.
• C. Castaing and M. Valadier: Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin, 1977
• A. Geletu: Introduction to topological spaces and set-valued maps (Lecture Notes).

Continuity...Existence of solution for CCOPT

Corollary (Existence of solution)

If f is a continuous function, $g : U \times \Omega \to \mathbb{R}$ is continuous, U is a compact set, and Ω is closed sets, then $p(u)$ is ***upper semi-continuous*** *and the feasible set $\mathcal{P} = \{u \in U \mid p(u) \geq \alpha\}$ is a compact set. Hence, the chance constraints optimization problem*

$$(CCOPT) \qquad \min_{u} f(u)$$
$$\text{subject to:}$$
$$Pr\{g(u,\xi) \leq 0\} \geq \alpha$$
$$u \in U$$

has a solution.

Proof.

Follows by the Weierstrass theorem: a continuous function attains its optimum (minimum or maximum) value on a closed and bounded set. □

Continuity...

Theorem (Raik 1971)

If the $g : U \times \Omega$ *are continuous and, for each* $u \in U \subset \mathbb{R}^m$,

$$Pr(\Gamma_0(u)) = 0, \quad \textbf{\textit{(measure-zero property)}}$$

where $\Gamma_0(u) = \{\xi \in \Omega \mid g_j(u, \xi) = 0\}$, *then probability function* $p(u) = Pr\{g(u, \xi) \leq 0\}$ *is continuous; hence, the feasible set of CCOPT*

$$\mathcal{P} = \{u \in U \subset \mathbb{R}^p \mid p(u) \geq \alpha\}$$

is a closed set.

Hint: Proofs of the above continuity statements need the following concepts:
- convergence in distribution
- convergence in probability
- almost sure convergence
- convergence in mean (square)
properties random variables.

Convexity

Proposition

Suppose $\xi = a^\top = (a_1, a_2, \ldots, a_n)$ be normally distritbuted (not necessarily independent) random variables with mean μ and covariance matrix Σ. If $p(u) = Pr\{a^\top u \leq b\} \geq \alpha$ and $b \in \mathbb{R}$, then the feasible set

$$\mathcal{P} = \{u \in \mathbb{R}^m \mid p(u) \geq \alpha\},$$

is convex for any $\alpha \in [0.5, 1]$.

Proof.

Since $p(u) \geq \alpha$ is equivalent to $\mu^\top u + \Phi^{-1}(\alpha)\sqrt{u^\top \Sigma u} \leq b$, the feasible set $\mathcal{P}$ is convex if the function $g(u) := \mu^\top u + \Phi^{-1}(\alpha)\sqrt{u^\top \Sigma u}$ is convex. But g is a convex w.r.t. u if $\Phi^{-1}(\alpha) \geq 0$ and this holds true due to the assumption that $\alpha \in [0.5, 1]$.

Convexity...

Generally, the convexity of the probability function

$$p(u) = Pr\{g(u,\xi) \leq 0\}$$

depends on

(a) convexity properties of the function $g(u,\xi)$

(b) some convexity property the pdf $\phi(\xi)$ of the random variable ξ.

A Motivational Example:

Let $g(u,\xi) = \xi - r(u)$ and ξ be a scalar random variable with distribution function Φ.

Let

$$p(u) = Pr\{\xi - r(u) \leq 0\},$$

where $r(u)$ is some function. Then, we have

$$p(u) = Pr\{\xi - r(u) \leq 0\} = Pr\{\xi \leq r(u)\} = \Phi(r(u))$$

If

- $r(u)$ is concave function
- Φ is such that $\Phi > 0$, non-decreasing and a log-concave function

the $p(u) = \Phi(r(u))$ is a log-concave function; hence, the feasible set

$$\mathcal{P} = \{u \in \mathbb{R}^m \mid p(u) \geq \alpha\}$$

is a convex set.

Convexity...A Motivational Example...

We use the second derivative test on the function $\log(\Phi(r(u)))$.
Hence the second derivative can be written as

$$\nabla^2 \left[\log(\Phi(r(u)))\right] = \frac{1}{[\Phi(r(u))]^2}\left[\underbrace{\Phi(r(u))\cdot\Phi''(r(u)) - \left[\Phi'(r(u))\right]^2}_{\leq 0}\right]\nabla r(u)\nabla r(u)^\top + \frac{1}{[\Phi(r(u))]}\underbrace{\Phi'(u)}_{>0}\nabla^2 r(u)$$

The matrix $\nabla r(u)\nabla r(u)^\top$ is (a rank 1) positive definite matrix; while $\nabla^2 r(u)$ is a negative definite matrix.

Hence, $\nabla^2\left[\log(\Phi(r(u)))\right]$ is a negative semi-definite matrix. Consequently, $p(u) = (\Phi(r(u)))$ is a log-concave function.

Convexity...

Definition (quasi-concave probability measure)

A probability measure $Pr(\cdot)$ is said to be quasi-concave if

$$Pr\left(\lambda S_1 + (1-\lambda) S_2\right) \geq \min\left(Pr(S_1), Pr(S_2)\right)$$

for all convex measurable sets S_1, S_2 and all $\lambda \in [0,1]$.

Theorem (Wets 1989)

If $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is a (jointly) convex function w.r.t. (u, ξ) and the probability measure $Pr(\cdot)$ is quasi-concave, then the feasible set

$$\mathcal{P} = \{u \in \mathbb{R}^n \mid Pr\{g(u,\xi) \leq 0\} \geq \alpha\}$$

is a convex set for all $\alpha \in [0,1]$.

Convexity...

Proof.

Given α, let $u_1, u_2 \in \mathcal{P}$ and $\lambda \in [0,1]$.
WTS: $\lambda u_1 + (1-\lambda)u_2 \in \mathcal{P}$.
Define the set

$$S(u) = \{\xi \in \mathbb{R}^p \mid g(u,\xi) \leq 0\}.$$

Then the sets $S(u_1)$ and $S(u_2)$ are convex and measurable and $Pr(S(u_1)) \geq \alpha$ and $S(u_2) \geq \alpha$.

For any $\xi_1 \in S(u_1)$ and $\xi_2 \in S(u_2)$ we have

$$g(\lambda(u_1,\xi_1) + (1-\lambda)(u_2,\xi_2)) \leq \lambda g(u_1,\xi_1) + (1-\lambda)g(u_2,\xi_2) \leq 0.$$

Set $u_\lambda := \lambda u_1 + (1-\lambda)u_2$ and $\xi_\lambda := \lambda\xi_1 + (1-\lambda)\xi_2$. It follows that $\xi_\lambda \in S(u_\lambda)$.

$$\Rightarrow \quad \lambda\xi_1 + (1-\lambda)\xi_2 \in S(\lambda u_1 + (1-\lambda)u_2), \forall \xi_1 \in S(u_1), \forall \xi_2 \in S(u_2)$$
$$\Rightarrow \quad \lambda S(u_1) + (1-\lambda)S(u_2) \subset S(\lambda u_1 + (1-\lambda)u_2)$$
$$\Rightarrow \quad Pr\left(S(u_\lambda)\right) \geq Pr\left(\lambda S(u_1) + (1-\lambda)S(u_2)\right) \geq \min\left\{Pr(S(u_1)), Pr(S(u_2))\right\} \geq \alpha.$$

Convexity...

Corollary

If $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is a (jointly) convex function w.r.t. (u, ξ) and the probability measure $Pr(\cdot)$ is quasi-concave, then the probability function

$$p(u) = \; Pr\{g(u, \xi) \leq 0\}.$$

is quasi-concave.

Remark:

(i) The result of the above theorem can be extended to

$$Pr\{g_j(u, \xi) \leq 0, j = 1, \ldots, q\} \qquad \text{(joint chance constraints)}$$

(ii) If

$$g(u, \xi) = Au + \xi,$$

then g is jointly convex w.r.t. (u, ξ).

Convexity...

Definition (log-concave measures, Prekopa 1971)

A probability measure $Pr(\cdot)$ is said to be log-concave if

$$Pr\left(\lambda S_1 + (1-\lambda)S_2\right) \geq \left[Pr(S_1)\right]^{\lambda} \cdot \left[Pr(S_2)\right]^{1-\lambda}$$

for all convex measurable sets S_1, S_2 and all $\lambda \in [0,1]$.

Proposition

A log-concave probability measure is quasi-concave.

Note that: For $0 \leq a, b (\leq 1)$ we have

$$a^{\lambda} b^{1-\lambda} \geq \left[\min(a,b)\right]^{\lambda} \cdot \left[\min(a,b)\right]^{1-\lambda} = min(a,b).$$

Convexity...

Proposition

Suppose $Pr(\cdot)$ be a probability measure associated with a density function $\phi(\xi)$ of a random variable $\xi \in \mathbb{R}^p$ such that

$$Pr(A) = \int_A \phi(\xi) d\xi.$$

for any measurable set A. If **ϕ is a log-concave function**, then $Pr(\cdot)$ is a log-concave probability measure.

Examples:

- If $Q(\xi)$ is a convex function, then any density function of the form $\phi(\xi) = e^{-Q(\xi)}$ is log-concave.
- The normal distribution has a log-concave density function.
- Other log-concave distributions: uniform, beta, gamma, Dirichlet, etc.

Differentiability

(A) A chance constraint

$$p(u) = Pr\left\{\xi \leq u\right\} = \Phi(u_1, \ldots, u_m)$$

where ξ is a vector of **independent random variables**; i.e. $\phi(\xi) = \Pi_{i=1}^{m} \phi_j(\xi_j)$.
Thus,

$$p(u) = \int_{-\infty}^{u_1} \ldots \int_{-\infty}^{u_m} \phi(\xi) d\xi_p \ldots d\xi_1 = \int_{-\infty}^{u_1} \ldots \int_{-\infty}^{u_m} \Pi_{i=1}^{m} \phi_j(\xi_j) d\xi_m \ldots$$

$$\frac{\partial p}{\partial u_i} p(u) = \phi_i(u_i) \times \left(\int_{-\infty}^{u_1} \ldots \int_{-\infty}^{u_{i-1}} \int_{-\infty}^{u_{i+1}} \int_{-\infty}^{u_m} \Pi_{\substack{i=1 \\ j \neq i}}^{m} \phi_j(\xi_j) d\xi_m \ldots d\xi_{i-1} d\xi_{i+1} \ldots d \right.$$

Differentiability...

(B) A simple chance constraint with correlated random variables

Theorem (Prekopa 1995)

If $\xi \sim \mathcal{N}(\mu, \Sigma)$ *with a positive definite covariance matrix* $\Sigma \in \mathbb{R}^{m \times m}$*, then probability function*

$$p(u) = Pr\{\xi \leq u\} = \Phi(u_1, \ldots, u_m)$$

is continuously differentiable and

$$\frac{\partial p}{\partial u_i} p(u) = \phi_i(u_i) \times \Phi(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_m),$$

where ϕ_i *is the one-dimensional Gaussian density function of* ξ_i *and* $\Phi(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_m)$ *is the normal distribution function of* $\widetilde{\xi} = (\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_m)$.

Proof idea: Use the Cholesky decomposition $\Sigma = LL^\top$, where L is an invertible lower-triangular matrix and define the transformation $\eta = L^{-1}(\xi - \mu)$. Then η has a standard normal distribution.

Differentiability ... a special case

(C) A simple generalization of Prekopa (Henrion and Möller 2012)

$$p(u) = Pr\left\{A\xi \leq u\right\}$$

with $A \in \mathbb{R}^{m \times p}$ and $\xi \sim \mathcal{N}(\mu, \Sigma)$ with a positive definite covariance matrix $\Sigma \in \mathbb{R}^{m \times m}$. Then $p(u)$ is continuously differentiable.

Note that:

$$\xi \sim \mathcal{N}(\mu, \Sigma) \Rightarrow \eta := A\xi \sim \mathcal{N}\left(A\mu, A\Sigma A^\top\right).$$

Differentiability...

(D) A more general nonlinear chance constraint

$$p(u) = Pr\left\{g(u,\xi) \leq 0\right\}.$$

Assumption-D

For each $u \in \mathbb{R}^m$, $\nabla_\xi g(u,\xi) \neq 0$ for each ξ on the boundary of the set $S(u) := \{\xi \in \Omega \mid g(u,\xi) \leq 0\}$.

Define

$$\begin{aligned} \partial S(u) &:= \text{boundary of } S(u) = \{\xi \in \Omega \mid g(u,\xi) \leq 0\} \\ \Gamma_0(u) &:= \{\xi \in \Omega \mid g(u,\xi) = 0\}. \end{aligned}$$

Then it follows that

$$\Gamma_0(u) \subset \partial S(u).$$

Differentiability...

Theorem (Uryasev 1995 , Marti 1996)

Given $u \in \mathbb{R}^m$. Suppose Assumption-D is satisfied. If

(i) *the function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ has continuous partial derivatives $\nabla_u g(u, \xi)$ and $\nabla_\xi g(u, \xi)$;*

(ii) *the random set $S(\cdot)$ is bounded in a neighborhood of u, then*

  (a) *the probability function $p(\cdot)$ is differentiable at u and*

  (b) *its gradient is given by*

$$\nabla p(u) = -\int_{\partial S(u)} \left(\frac{\phi(\xi)}{\|\nabla_\xi g(u, \xi)\|} \nabla_u g(u, \xi) \right) dS$$

- This formula for $\nabla p(u)$ is less practical in computational methods.

Major difficulties in CCOPT

Major difficulties

- Generally, there is usually no closed-form analytic (deterministic) representation for $p(u)$
- Amenable (continuity and convexity) structures of $p(u)$ are available only for **Gaussian distributed** ξ.
- The $p(u)$ probability function is, generally, difficult to evaluate directly (intractability).
  $\Rightarrow$ numerical or analytical approximation methods
- The function $p(u)$ can be a **non-differentiable function** $\Rightarrow$ smoothing approximations or non-smooth analysis strategies

- CCOPT is a hard optimization problem.

Recommended Approach

Use approximation methods.

Approximation methods should:

- facilitate tractability of chance constraints
- guarantee a-priori feasibility
- enable the consideration of general probability distributions for ξ
- etc

Widely used Solution Methods for CCOPT

- Linearization (Garnier et al. 2008)
- Back-mapping (Wendt *et al.* 2002, Geletu *et al.* 2011)
- Sample average approximation (Shapiro 2003, Pagnoncelli et al. 2009)
- Robust (Semi-infinite) optimization approach (Califore & Campi 2005)
- Analytic approximation (Nemirovski & Shapiro, Geletu *et al.*)

Approximation methods

(I) **Linearization** (Garnier et al. 2008)

Assumptions

- $\xi^\top = (\xi_1, \ldots, \xi_p)$ a vector of normally distributed random variables with mean $\mu = 0$ and covariance matrix Σ;
- for each fixed u, $g(u, \cdot)$ is at least twice differentiable w.r.t. ξ.

For ξ having a sufficiently small variance we have

$$g(u,\xi) \approx g(u,0) + \sum_{j=1}^{p} \frac{\partial g_j}{\partial \xi_j}(u,0)\xi_j = a_0(u) + \xi_1 a_1(u) + \ldots + \xi_p a_p(u).$$

- This could provide an analytic approximation
- But works only when the variance of ξ is very small;
- If ξ is non-Gaussian, tractability is still an issue (Nemirovski 2012) .

Approximation methods ...

(II) **Back-Mapping** (Wendt et al. 2002)

Idea of back-mapping

- Find a monotonic relation between $Z = g(u, \xi)$ and some random variable ξ_j.
- That is, verify theoretically (see Geletu et al. 2011) or experimentally that there is a real valued function φ such that, for any $u \in U$
  - $Z = \varphi_u(\xi_j)$;
  - φ_u is strictly increasing ($\xi_j \uparrow Z$) or decreasing $\xi_j \downarrow Z$)

$\Rightarrow$ $\xi_j = \varphi_u^{-1}(Z)$. Furhtermore

- $\xi_j \uparrow Z \Rightarrow Pr\{g(u, \xi) \leq 0\} = Pr\{\xi_j \leq \varphi_u^{-1}(0)\}$.
- $\xi_j \downarrow Z \Rightarrow Pr\{g(u, \xi) \leq 0\} = Pr\{\xi_j \geq \varphi_u^{-1}(0)\}$.

Reference

- A. Geletu *et al.*: Monotony analysis and sparse-grid integration for nonlinear chance-constrained chemical process optimizationa problems. *Engineering Optimization*, 43(10): 1019–1041, 2011.

Approximation methods - Back Mapping...

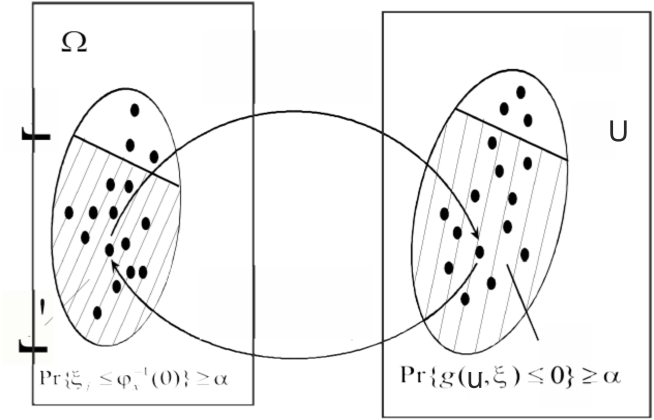


Figure: Back projection of chance constraints

Back-mapping (, Geletu et al. 2011):

- usable only if certain monotonic relations hold true

Approximation methods - Back Mapping...

- Hence, for $\xi_j \uparrow Z$, CCOPT is equivalent to

$$(CCOPT) \qquad \min_u E\left[f(u,\xi)\right]$$
$$\text{subject to:}$$
$$p(u) = Pr\{\xi \in \Omega \mid \xi_j \leq \varphi_u^{-1}(u)\} \geq \alpha,$$
$$u \in U.$$

Now, e.g. assuming $\Omega = \mathbb{R}^p$ and $\phi(\xi) = \Pi_{i=1}^p \phi_i(\xi_i)$, we have

$$p(u) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \int_{-\infty}^{\varphi_u^{-1}(u)} \phi(\xi) d\xi \qquad (28)$$

$$\nabla p(u) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{\text{p-1 integrals}} \phi_j(\varphi_u^{-1}(u)) \nabla \varphi_u^{-1}(u) \Pi^p_{\substack{i=1 \\ i \neq j}} \phi_i(\xi_i) d\xi \qquad (29)$$

▶ However, back-mapping is usable only if monotonic relations are easy to identify. But monotonic relations can be difficult to verify.

Approximation methods ... Robust optimization

(III) **Robust optimization approach (Califore & Campi 2005)**
Robust optimization considers the (*worst-case*) problem

$$(RO) \qquad \min_u E\left[f(u,\xi)\right]$$

subject to:

$$g(u,\xi) \le 0, \xi \in \Omega,$$
$$u \in U,$$

where $g(u,\xi) \le 0$ is required to be satisfied as for many realizations of ξ from Ω as possible.

Randomized solution based on scenario generation:

- Generate independent identically distributed random samples $\xi^1, \ldots, \xi^N$ from Ω ((Monte-Carlo method)).

... Robust optimization approach ...

- Solve the optimization problem

$$(NLP)_{RO} \qquad \min_u \frac{1}{N}\sum_{k=1}^{N} f(u,\xi^k)$$
$$s.t. \quad g(u,\xi^k) \leq 0, k = 1,\dots,N;$$
$$u \in U.$$

Theorem (Califore & Campi 2005)

Suppose $\alpha \in (0,1)$ and $f(\cdot,\xi)$ is convex w.r.t. $u \in \mathbb{R}^m$. If the number of random samples $\xi^1,\dots,\xi^N$

$$N \geq \frac{2n}{(1-\alpha)} ln\left(\frac{1}{1-\alpha}\right) + \left(\frac{2}{1-\alpha}\right) ln\left(\frac{1}{\alpha}\right) + 2m,$$

then the optimal solution obtained from $(NLP)_{RO}$ is an optimal solution of (RO) with reliability α.

... Robust optimization approach ...

Example: According Califore & Campi
if m = 10 and = 0.9999; the number of required samples should be at least $N \approx 1,842,089$ to satisfy

$$Pr\{g(u,\xi) \leq 0\} \geq \alpha$$

with $\alpha = 0.9999$ for the optimal u^*.

Approximation methods ... Sample average approximation

(IV) **Sample Average Approximation (SAA) (Shapiro 2003, Pagnoncelli et al. 2009)**

- Define

$$\mathbb{I}_{(0,+\infty]}(G(u,\xi)) = \begin{cases} 0, & \text{if } g(u,\xi) > 0 \\ 1, & \text{if } g(u,\xi) \leq 0. \end{cases}$$

- Generate a sequence of deterministic points $\{\xi^1, \ldots, \xi^N\} \subset \Omega$ with *low discrepancy* property; e.g. Quasi-Monte-Carlo sequences like **Fourer**, **Sobol** or **Niederreiter** etc.

- Replace the chance constrains with

$$p_N(u) = \frac{1}{N} \sum_{k=1}^{N} \mathbb{I}_{(-\infty,0]}\left(g(u,\xi^k)\right) \geq \alpha.$$

This is a relative frequency-count approximation of chance constraints.

Approximation methods ... SAA ...

- Solve in stead of CCOPT the deterministic optimization problem

$$(NLP)_{SAA} \qquad \min_u \frac{1}{N}\sum_{k=1}^{N} f(u,\xi^k)$$

$$s.t. \quad \frac{1}{N}\sum_{k=1}^{N} \mathbb{I}_{(-\infty,0]}\left(g(u,\xi^k)\right) \geq \alpha$$

$$u \in U.$$

- SAA preserves convexity structures
- avoids the need to compute integrals
- feasibility of solution obtained from $(\text{NLP})_{SAA}$ to the CCOPT is guaranteed only for a very large sample-size N
- In RO and SAA methods, guaranteeing the feasibility of approximate optimal solutions is very expensive.
- **RO and SAA provide no a-priori guarantee for feasibility!**


"If you only do what you can do,
you'll NEVER BE MORE than
you are now."

Smoothing inner-outer approximation method Geletu et al. 2015, 2017

Objectives

- To develop analytic approximation methods for CCOPT with a general (non-convex) constraint function $g(x, \xi)$
- To consider both Gaussian and non-Gaussian continuous pdf $\phi(\xi)$
- To facilitate tractable solution of CCOPT problems for large-scale engineering applications

Smoothing inner-outer approximation ... Contributions

On Finite dimensional CCOPT

- A. Geletu, M. Klöppel, A. Hoffmann, P. Li, (2015). A tractable approximation of nonconvex chance constrained optimization with non-Gaussian uncertainties. Journal of Engineering Optimization, 47(4), pp. 495 - 520.
- A. Geletu , A. Hoffmann, M. Klöppel, P. Li, (2017). An inner-outer approximation approach to chance constrained optimization. SIAM Journal on Optimization, 27(3), 1834 - 1857.
- A. Geletu, A. Hoffmann, P. Li, (2019). Analytic approximation and differentiability of joint chance constraints. Optimization, 68(10), 1985–2023.

Infinite dimensional CCOPT

- A. Geletu, A. Hoffmann, P. Li, (2016). Chance constrained optimization on Banach spaces. 14th EUROPT Workshop on Advances in Continuous Optimization Warsaw (Poland), July 1-2, 2016. http://www.europt2016.ia.pw.edu.pl/schedule/FD-4.html
- A. Geletu, A. Hoffmann, P. Schmidt, P. Li, (2020). Smoothing methods to chance constrained optimization of elliptic PDE systems. ESAIM: COCV, 26 (2020), 1-28.

Smoothing inner-outer approximation ...

Equivalent representations

$$p(u) = Pr\{g(u,\xi) \leq 0\} \geq \alpha$$

$$\Leftrightarrow \quad \int_{g(u,\xi)\in[0,+\infty)} \phi(\xi)d\xi \geq \alpha$$

$$\Leftrightarrow \quad \mathbf{E}\left[\mathbf{h}(\mathbf{g}(\mathbf{u},\xi))\right] \leq \mathbf{1} - \alpha$$

$$h(s) := \begin{cases} 1, & \text{if } s \geq 0, \\ 0, & \text{if } s < 0, \end{cases}$$ (**Heaviside step function**).

Smoothing inner-outer approximation ...

Expected-value Representation

Define

$$h(u,\xi) := \begin{cases} 0, & \text{if } g(u,\xi) \le 0 \\ 1, & \text{if } g(u,\xi) > 0. \end{cases}$$

$\Rightarrow$

$$Pr\{g(u,\xi) > 0\} = E\left[h(u,\xi)\right]$$

$\Rightarrow$

$$Pr\{g(u,\xi) \le 0\} \ge \alpha \Leftrightarrow E\left[h(u,\xi)\right] \le 1 - \alpha.$$

An Equivalent Representation of CCOPT

$$(CCOPT) \qquad \min_{u} E\left[f\left(x,\xi\right)\right] \tag{30}$$

subject to:

$$E[h(x,\xi)] \leq 1-\alpha \tag{31}$$

$$x \in X. \tag{32}$$

Drawback

- $E\left[h(x,\xi)\right]$ can be discontinuous and hard to work-with.

Idea

- Design a smoothing approximation to $E\left[h(x,\xi)\right]$.

Smoothing Inner-outer approximation ...

Objectives

- to develop a **smoothing analytic approximation** to the probability function

$$(1 - p(u)) = E\left[h(g(u,\xi))\right]$$

- to guarantee
  - **(apriori) feasiblile points** to the CCOPT
  - approximate solution of CCOPT by **avoding the direction evaluation** of $p(u)$

Smoothing inner-outer approximation ...

Geletu-Hoffmann (GH) function

The parametric family (Geletu *et al*. 2015, 2017, 2020)

$$\Theta(\tau, s) = \frac{1 + m_1\tau}{1 + m_2\tau \exp\left(-\frac{s}{\tau}\right)}, \quad \textit{for} \quad \tau \in (0,1), \ s \in \mathbb{R}, \qquad (33)$$

satisfies properties P1-P5, where m_1, m_2 are constants with $0 < m_2 \leq m_2/(1 + m_1)$,. Define also, $\Pi(\tau, s) := \Theta(\tau, -s)$. Thus,

$$1 - \Theta(\tau, s) < h(-s) < \Theta(\tau, -s) = \Pi(\tau, s) \qquad (34)$$

Approximation functions

Define the functions (Geletu *et al*. 2015, 2017)

$$\psi(\tau, u) := E\left[\Theta(\tau, g(u, \xi))\right], \qquad (35)$$

$$\varphi(\tau, u) := E\left[\Pi(\tau, g(u, \xi))\right], \qquad (36)$$

where $\tau \in (0, 1)$.

Smoothing inner-outer approximation ... Problems

Inner-outer approximation problems

Inner Approximation	**Outer Approximation**
$(IA_\tau) \quad \min_u \; F(u)$ s.t. $\psi(\tau,u) \leq 1-\alpha,\; u \in U, \tau \in (0,1).$	$(OA_\tau) \quad \min_u \; F(u)$ s.t. $\varphi(\tau,u) \geq \alpha,\; u \in U, \tau \in (0,1).$

Respective feasible sets of IA_τ and OA_τ

$$\mathcal{M}(\tau) := \{u \in U \mid \psi(\tau,u) \leq 1-\alpha\}, \tau \in (0,1),$$

$$\mathcal{S}(\tau) := \{u \in U \mid \varphi(\tau,u) \geq \alpha\}, \tau \in (0,1).$$

Smoothing inner-outer approximation ... Properties

Geletu et al. 2015

Suppose $0 < \tau_2 \leq \tau_1 < 1$ and $g(\cdot, \xi)$ is continuous w.r.t. u. Then,

(1) ***monotonicity***: For $u \in U$,

$$\varphi(\tau_1, u) \geq \varphi(\tau_2, u) \geq p(u) \geq 1 - \psi(\tau_2, u) \geq 1 - \psi(\tau_1, u).$$

(2) ***smoothness***: $\psi(\tau, \cdot)$ and $\varphi(\tau, \cdot)$ are smooth if $g(\cdot, \xi)$ is smooth, for each fixed $\tau \in (0, 1)$.

(3) ***tight approximation***: For each $u \in U$,

$$p(u) = \inf_{\tau \in (0,1)} \varphi(\tau, u) \quad \text{and} \quad \sup_{\tau \in (0,1)} (1 - \psi(\tau, u)) = p(u), \tag{37}$$

Smoothing inner-outer approximation ... Properties

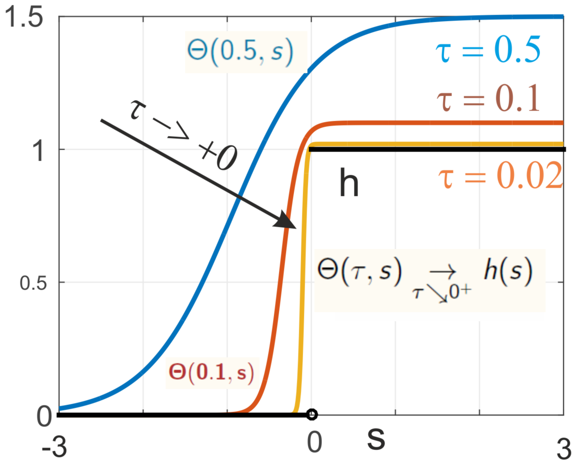


Figure: Convergence of Θ to h

Smoothing inner-outer approximation ... Properties...

Feasible set of CCOPT: $\mathcal{P} = \{u \in U \mid p(u) \geq \alpha\}$

(1) inner-outer approximation

$$\mathrm{M}(\tau) \subset \mathcal{P} \subset S(\tau), \text{ for any } \tau \in (0,1).$$

(2) monotonicity of the inner-outer approximations

$$\mathrm{M}(\tau_2) \subset \mathrm{M}(\tau_1) \subset \mathcal{P} \subset S(\tau_1) \subset S(\tau_2), \text{ for } 0 < \tau_1 < \tau_2 < 1.$$

(3) convergence of the feasible sets of the approximations

$$\lim_{\tau \searrow 0^+} \mathrm{M}(\tau) = \mathcal{P}, \lim_{\tau \searrow 0^+} S(\tau) = \mathcal{P},$$

(4) **Painlevé - Kuratowski convergence** (with $M_k := \mathcal{M}(\tau_k)$, $S_k := \mathcal{S}(\tau_k)$ for $\{\tau_k\}_{k \in \mathbb{N}} \subset (0,1)$ with $\tau_k \searrow 0^+$):

$$\lim_{k \to +\infty} M_k = \mathcal{P} \text{ and } \lim_{k \to +\infty} S_k = \mathcal{P}.$$

Smoothing inner-outer approximation ... Properties...

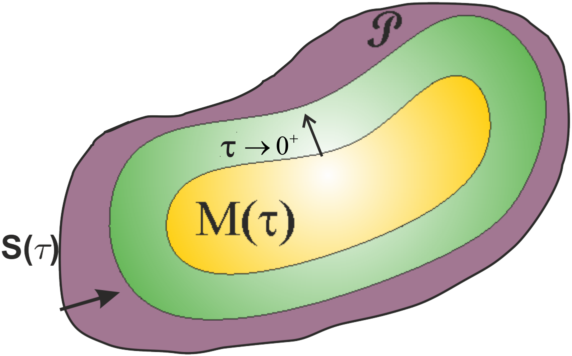


Figure: Inner-Outer approximation for the feasible set of CCOPT

Inner-outer approximation ...

Convergence of Gradients ...Geletu *et al.* 2015, 2017

(iii) If $g(\cdot,\xi)$ differentiable function, $\nabla_u g(u,\cdot)$ is Legesgue measurable w.r.t. ξ, and there is a Lebesgue measurable function $v : \Omega \to \mathbb{R}$ such that $\|\nabla_u g(u,\ \xi)\| \leq v(\xi)$, almost surely for $\xi \in \Omega$, then the function $\psi(\tau,\cdot)$ and $\varphi(\tau,\cdot)$ are differentiable w.r.t. u and

$$\begin{aligned}\nabla(1-\psi(\tau,u)) &= -\int_{\Omega} \frac{\partial\Theta(\tau,s)}{\partial s}\bigg|_{s=g(u,\xi)} \nabla_u g(u,\xi)\phi(\xi)d\xi,\\ \nabla\varphi(\tau,u) &= -\int_{\Omega} \frac{\partial\Theta(\tau,s)}{\partial s}\bigg|_{s=-g(u,\xi)} \nabla_u g(u,\xi)\phi(\xi)d\xi.\end{aligned}$$

Note that:

- $\psi(\tau,u)$ and $\varphi(\tau,u)$ are differentiable irrespective of the differentiability of $p(\cdot)$
  $\Longrightarrow$ IA_τ and OA_τ are **smoothing approximations** of CCOPT.
- The formulae for $\nabla(1-\psi(\tau,u))$ and $\nabla\varphi(\tau,u)$ are **simple to use than those by Uryasev and Marti**.

Inner-outer approximation ...

Properties ...Geletu *et al.*

(iv) If the same assumptions as in Marti 1996, Uryase 1995 hold true, $g(\cdot, \xi)$ and $p(\cdot)$ is a differentiable, then

$$ {}_{\tau \searrow 0^+} \nabla \psi(\tau, u) = -\nabla p(u) \text{ and } \lim_{\tau \searrow 0^+} \nabla \varphi(\tau, u) = \nabla p(u). $$

(iv) Let $\{\tau_k\}_{k \in \mathbb{N}}$ is any sequence of parameters such that $\tau_k \searrow 0^+$ and $\{u^*_{\tau_k}\}_{k \in \mathbb{N}}$ is the corresponding sequence of optimal solutions of $(\mathrm{IA})_{\tau_k}$ (or of $(\mathrm{OA})_{\tau_k}$). Then any limit point of $\{u^*_{\tau_k}\}_{k \in \mathbb{N}}$ is an optimal solution of CCOPT.

Inner-outer approximation ...

Algorithm 1: Conceptual inner-outer approximation method

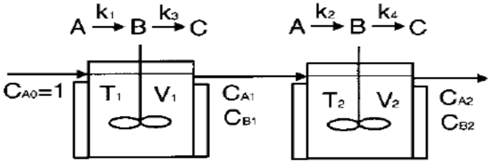
Choose an initial parameter $\tau_0 \in (0,1)$;
Solve the optimization problems $(\text{IA})_{\tau_0}$ and $(\text{OA})_{\tau_0}$;
Select the termination tolerance *tol*;
Set $k \leftarrow 0$
while $\left(\left| f_{IA}(u^*_{\tau_k}) - f_{OA}(\widehat{u}^*_{\tau_k}) \right| > tol\right)$ **do**
    Reduce the parameter τ_k (e.g., $\tau_{k+1} = \rho \tau_k$, for $\rho \in (0,1)$);
    Set $k \leftarrow k + 1$;
    Solve the optimization problems $(\text{IA})_{\tau_k}$ and $(\text{OA})_{\tau_k}$;
end while

Here

- $u^*_{\tau_k}$ and $\widehat{u}^*_{\tau_k}$ are optimal solution of $(\mathrm{IA})_{\tau_k}$ and $(\mathrm{OA})_{\tau_k}$, resp.
- $f_{IA}(u^*_{\tau_k})$; $f_{OA}(\widehat{u}_{\tau_k}*)$ are optimal objective values
- The algorithm terminates when the optimal values of the inner and outer approximations are almost equal.

5. Example: An engineering problem

Consider the reactor network design problem (Wendt *et al.* 2002)



- C_{A_j}, C_{B_j}, V_i, and $T_\mathrm{i}(i = 1, 2)$ are the concentration of the components of A and B, the volumes and temperatures of both reactors, respectively.
- kinetic parameters (the activation energy and the frequency factor in the Arrhenius equation) are uncertain.

Objective: To determine the minimum cost design strategies guaranteeing high reliability of satisfaction of product specifications.

An engineering problem ...

By defining $u_i = V_i, x_i = C_{A_i}, x_{i+2} = C_{B_i}$, for $i = 1, 2$,

$$(CCOPT) \qquad \min_{u \in \mathbb{R}^2} \{ f(u) = \sqrt{u_1} + \sqrt{u_2} \} \tag{38}$$

subject to:

$$x_1 + k_1(\xi) x_1 u_1 = 1, \tag{39}$$

$$x_2 - x_1 + k_2(\xi) x_2 u_2 = 0, \tag{40}$$

$$x_3 + x_1 + k_3(\xi) x_3 u_1 = 1, \tag{41}$$

$$x_4 - x_3 + x_2 - x_1 + k_4(\xi) x_4 u_2 = 0, \tag{42}$$

$$Pr\{x_4 \geq x_{\min}\} \geq \alpha, \tag{43}$$

$$0 \leq u_1 \leq 16, \; 0 \leq u_2 \leq 16, \tag{44}$$

$$\begin{aligned} k_1(\xi) &= \xi_1 \exp(-\xi_3/RT_1), \quad k_2(\xi) = \xi_1 \exp(-\xi_3/RT_2) \\ k_3(\xi) &= \xi_2 \exp(-\xi_4/RT_1), \quad k_4(\xi) = \xi_4 \exp(-\xi_2/RT_2), \\ \alpha &\in [0.5, 1], RT_1 = 5180.869 \text{ and } RT_2 = 4765.169. \end{aligned}$$

An engineering problem ... Compact form

By solving the model equation (39)-(42) we obtain

$$x_4(u,\xi) = \frac{k_2u_2(1+k_1u_1+k_3u_1)+k_1u_1}{(1+k_1u_1)(1+k_2u_2)(1+k_3u_3)(1+k_4u_4)}.$$

Hence,

$$(CCOPT) \qquad \min_{u\in\mathbb{R}^2}\{f(u)=\sqrt{u_1}+\sqrt{u_2}\} \tag{45}$$

subject to:

$$Pr\{x_4(u,\xi)\geq x_{\text{min}}\}\geq\alpha, \tag{46}$$

$$0\leq u_1\leq 16,\ 0\leq u_2\leq 16, \tag{47}$$

Note that: $g(u,\xi) = -x_4(u,\xi) + x_{min}$

An engineering problem ....Inner-outer approximation

$$(CCOPT) \qquad \min_{u\in\mathbb{R}^2}\{f(u)=\sqrt{u_1}+\sqrt{u_2}\}$$

subject to:

$$Pr\{-x_4(u,\xi)+x_{\min}\le 0\}\ge\alpha,$$

$$0\le u_1\le 16,\ \ 0\le u_2\le 16,$$

Approximation functions

$$\psi(\tau,u)=E[\Theta_{gh}(\tau,-x_4(u,\xi)+x_{\min})]=E\left[\frac{1+m_1\tau}{1+m_2\tau\exp(-\frac{1}{\tau}(-x_4(u,\xi)+x_{\min}))}\right.$$

$$\varphi(\tau,u)=E[\Theta_{gh}(\tau,x_4(u,\xi)-x_{\min})]=E\left[\frac{1+m_1\tau}{1+m_2\tau\exp(\frac{1}{\tau}(-x_4(u,\xi)+x_{\min}))}\right]$$

A numerical example ...

Data:

$$k_1(\xi) = \xi_1 \exp(-\xi_3/RT_1), \quad k_2(\xi) = \xi_1 \exp(-\xi_3/RT_2)$$
$$k_3(\xi) = \xi_2 \exp(-\xi_4/RT_1), \quad k_4(\xi) = \xi_4 \exp(-\xi_2/RT_2),$$

$\alpha = 0.9$, $RT_1 = 5180.869$ and $RT_2 = 4765.169$.

	Expected value	Standard deviation	Correlation matrix
ξ_1	0.715	0.0215	$\begin{bmatrix} 1 & 0.5 & 0.3 & 0.2 \\ 0.5 & 1 & 0.5 & 0.1 \\ 0.3 & 0.5 & 1 & 0.3 \\ 0.2 & 0.1 & 0.3 & 1 \end{bmatrix}$
ξ_2	0.182	0.0055	
ξ_3	6665.948	200	
ξ_4	7965.248	240	

Table: Mean, standard deviation and correlation matrix of the random variables

A numerical example ...

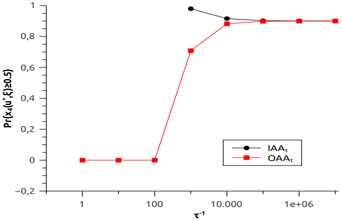


Figure: Optimal objective function values of IA_τ and OA_τ for decreasing values of τ.

- IA_τ and OA_τ are solved using IpOpt.
- Integrals are evaluated using quasi-Monte Carlo samples.
- For $\tau = 10^{-7}$,

$$Pr\left\{x_4(u^*_{IA_\tau}, \xi) \geq 0.5\right\} - Pr\left\{x_4(u^*_{OA_\tau}, \xi) \geq 0.5\right\} \approx 4.8 \times 10^{-5}.$$

Inner-outer approximation ...

Advantages:

- A nonsmooth CCOPT can be approximately solved by solving smooth nonlinear optimization problems
- The problems IA_τ and OA_τ can be easily soleved by a gradient-based algorithm
- The problem IA_τ guarantees an a-priori feasible approximate solution to CCOPT
- The inner-outer approximation can be used irrespective of the distribution of ξ

Disadvantages

- Requires intensive computations due to the need to evaluate multi-dimensional integrals
- The choice of τ should balance tighter analytic approximation and computational accuracy

1. Chance constrained optimization on Banach spaces

$$(CCBS) \qquad \min_u E\left[J(u,y,\xi)\right] \tag{48}$$

subject to:

$$\mathcal{A}(u,y,\xi) + BC = 0, \tag{49}$$

$$Pr\left\{g(u,y,\xi) \leq 0\right\} \geq \alpha, \tag{50}$$

$$u_{min} \leq u \leq u_{max} \tag{51}$$

$$u \in E, \tag{52}$$

where $\mathcal{A}$ is an operate acting on the behavior y of the system on a Banach space, E is a reflexive Banach space.

2. Chance constrained mixed integer problems

(CCPDE)

$$\min_u E\left[\|y - y_d\|^2_{H_0^1(D)}\right] + \frac{\rho}{2}\|u\|^2_{\mathcal{L}^2(D)}$$

subject to:

$$-\nabla(\kappa(x,\xi_0)\nabla y) = \sum_{k=1}^{p} f_k(u,x,\xi_k), \text{ in } D \text{ a.e. } \Omega,$$

$$y|_{\partial D} = g(x,\xi_{p+1}), \text{ a.e. } \Omega,$$

$$Pr\left\{y_{min} \le y \le y_{max} \le 0\right\} \ge \alpha,$$

$$u \in \mathcal{U} = \{u \in L^2(D) \mid u_{min} \le u \le u_{max}\},$$

The variable $\xi^\top = (\xi_0, \xi_1, \ldots, \xi_p, \xi_{p+1})$ is random.

Ref: A. Geletu, A. Hoffmann, P. Schmidt, P.Li.: Chance constrained optimization of elliptic PDE systems with a smoothing convex approximation. ESAIM: COCV, Volume 26, 2020.

3. Chance constrained model predictive controller of a semilinear parabolic PDE system

(CCMPC)

$$\min_{u}\left\{J(u)=E\left[\int_{\hat{t}_z}^{\hat{t}_z+H}\left\{\|y(u,\xi;t,\cdot)-y_d(t,\cdot)\|^2_{L^2(D)}+\frac{\lambda}{2}\|u(t,\cdot)\|^2_{L^2(D)}\right\}dt\right]\right\}$$

subjected to:

$$\frac{\partial y}{\partial t}-\nabla_x\left[\kappa(x,\xi)\nabla_x y\right]=f(u,x,t) \text{ in } Q\times\Omega,$$

$$-\kappa_0\nabla y\cdot\mathbf{n}=g(y,t,x) \text{ on } (\hat{t},\hat{t}+H]\times\partial D\times\Omega,$$

$$y(\hat{t}_z,x)=y_{\hat{t}_z}(x) \text{ in } D,$$

$$Pr\left\{y(u,\xi;t,x)\leq y_{max}\right\}\geq\alpha, \text{ in } Q,$$

$$u_{min}\leq u(t,x)\leq u_{max}, \text{ in } Q,$$

$$\hat{t}_{z+1}:=\hat{t}_{z+1}+\Delta t, z=1,\ldots,N'_t,$$

where

- $D\subset\mathbb{R}^n$ $(n=1,2,3)$
- $(\hat{t}_z,\hat{t}_z+H]$ is a prediction time-horizon of finite length H
- $Q:=(\hat{t}_z,\hat{t}_z+H]\times D$;

8. Current research topics ...

Fault-tolerance and safety tubes (corridors) for stochastic MPC.

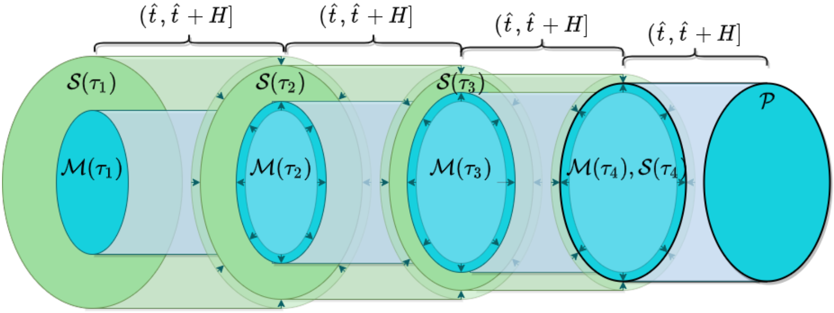


Figure: Deterministic tubes generated through the inner- outer approximation over prediction horizons, for decreasing values of the approximation parameter $\tau_1 > \tau_2 > \tau_3 > \tau_4$.

Ref:R. Voropai, A. Geletu, P. Li: Model predictive control of parabolic PDE systems under chance constraints. (to be submitted)

4. Chance constrained mixed integer nonlinear programming (MINLP) problems

$$(CCMINLP) \qquad \min_{u,z} E\left[J(u,z,y,\xi)\right] \qquad (53)$$

subject to:

$$G(u,z,y,\xi) = 0, \qquad (54)$$

$$Pr\left\{g(u,z,y,\xi) \le 0\right\} \ge \alpha, \qquad (55)$$

$$u_{min} \le u \le u_{max} \qquad (56)$$

$$z \in \{0,1\}^q, u \in \mathbb{R}^m \qquad (57)$$

Ref: A. Tesfaye, A. Geletu, B. Guta: Chance Constrained Mixed-Integer Nonlinear Programming and Applications.

(in progress)

Open issues

- Since $p(u)$ is generally not differentiable
  1. what are the convenient sub-differential characterizations of $p(u)$ (Clark subdifferentials, Mordukhovich subdifferentials, Frechet subdifferential, etc.)
  2. what are the relationships between a subdifferential of $\partial p(u)$ and the set of gradients $\{\nabla\psi(\tau, u) | \tau \in (0, 1)\}$ of the smoothing function (resp. $\{\nabla\varphi(\tau, u) | \tau \in (0, 1)\}$)
- Convex inner-outer approximation
  1. If (CCOPT) is a convex function, then the outer approximation $(\mathrm{OA})_\tau$ is convex. (Geletu *et al.*, 2020)
  2. However, the inner approximation $(\mathrm{IA})_\tau$ with the GH function may not be convex, even if (CCOPT) is a convex problem.

Idea: (i) Design (a new) or use another function $\Theta(\tau, s)$ for the inner and outer approximation.

References

- Ben-Tal, A., El Ghaoui, L., Nemirovski, A., (2009). Robust Optimization. Princeton University Press, Woodstock, United Kingdom, 2009.
- Ben-Tal, A., Nemirovski, A., (2009). On safe tractable approximations of chance constrained linear matrix inequalities, J. Math. Oper. Res., 34(1), 1–25.
- Ben-Tal, A., Nemirovski, A., (1999). Robust solutions of uncertain linear programs. OR Letters, 25(1), 1–13.
- Ben-Tal, A., Nemirovski, A., (1998). Robust convex optimization. Mathematics of Operations Research, 23(4), 769–805.
- Bertsimas, D., David, D.B.,, Caramanis, C., (2011). Theory and applications of robust optimization, SIAM Rev., 53 (2011), pp. 464–501.
- Charnes, A., and Cooper, W. (1959). Chance constrained programming. Management Science, 6, 73–79.
- Charnes, A., Cooper, W., and Symmonds, G.H. (1958). Cost horizons and certainty equivalence: an approach to stochastic programming of heating oil. Management Science, 4, 235–263.
- Garnier, J., Omrane, A., Rouchdy, Y. 2009. Asymptotic formulas for the derivatives of probability functions and their Monte Carlo estimations. *European Journal of Operations Research*, 198(3): 848–858.
- Geletu, A., Hoffmann, A., Klöppel, M., Li, P., 2017. An inner-outer approximation approach to chance constrained optimization. SIAM Journal on Optimization, 27(3), 1834 - 1857.
- Geletu, A., Klöppel, M., Hoffmann, A., Li, P., 2015. A tractable approximation of nonconvex chance constrained optimization with non-Gaussian uncertainties. *Journal of Engineering Optimization*, 47(4), pp. 495 - 520.

References

- Geletu, A., Klöppel, M., Zhang, H., Li, P., 2012. Advances and applications of chance-constrained approaches to systems optimization under uncertainty. *International Journal of Systems Science*, 44(7): 1209–1232.
- Geletu, A., Li, P., 2014. Recent developments in computational approaches to optimization under uncertainty and application in process systems engineering, *ChemBioEng Reviews*, 1(4), 170–190.
- Geletu, A., Hoffmann, A., Klöppel, M., Li, P. 2011. Monotony analysis and sparse-grid integration for nonlinear chance-constrained chemical process optimizationa problems. *Engineering Optimization*, 43(10): 1019–1041.
- Henrion, R., Strugarek, C., (2008). Convexity of chance constraints with independent random variables. *Computational Optimization and Applications*, 41(2): 263–276.
- Hong, J. L., Yang, Y., Zhang, L. (2011). Sequential convex approximations to joint chance constrained programs: A Monte Carlo approach. *Operations Research*, 59(3): 617–630.
- Kall, P., and Wallace, S.W., Stochastic Programming. Wiley & Sons, Chichester, England, 1995.
- Kibzun, A.I., Kan, Y.S., Stochastic Programming Problems: With Probabbility and Quantile Functions. Wiley & Sons, Chichester, England, 1996.
- Klöppel, M., Geletu, A., Hoffmann, A., Li, P., (2011). Using sparse-grid methods to improve computation efficiency in solving dynamic nonlinear chance-constrained optimization problems. Industrial Engineering Chemical Research, 50, 5693-5704.
- Marti, K. 1996. Differentiation formulas for probability functions: the transformation method. *Mathematical Programming*, 75(2): 201–220.

References...

- Miller, B. L., Wagner, H. M., (1965). Chance-constrained programming with joint constraints. Operations Research, 13(6), 930–945
- Nemirovski, A., Shapiro, A. 2006. Convex approximation of chance constrained programs. SIAM Journal on Optimization, 17(4): 969-996.
- Pagnoncelli, B. K., Ahmed, S., Shapiro, A. 2009. Sample average approximation method for chance constrained programming: theory and applications. *Journal of Optimization Theory and Applications*, 142(2): 399–416.
- Norkin, V. I., 1993. The analysis and optimization of probability functions. Working Paper, WP-93-6, International Institute for Applied Systems Analysis, A-2361. Laxenburg: Austria.
- Pinter, J. 1989. Deterministic approximation of probability inequalities. *Operations Research*, 33(4); 219–239.
- Prekopa, A. 1995. *Stochastic programming*. Dordrecht: Kluwer Academic.
- Raik, E., (1971). The qualtile functions in stochatic nonlinear programming. Eesti Nsv Teaduste Akadeemia Toimetised (News of Estonian Academy of Sciences), Füüsika-Matemaatika, 20, 229–231.[in Russian]
- Raik, E., (1971). The quantile functions in stochatic nonlinear programming. Eesti Nsv Teaduste Akadeemia Toimetised (News of Estonian Academy of Sciences), Füüsika-Matemaatika, 20, 229–231.[in Russian]
- Rockafellar, R. T., Uryasev, S. 2000. Optimization of conditional value-at-risk. *Journal of Risk*, 2(4), 21–51.

References...

- Römisch, W., Schultz, R., (1991). Distribution sensitivity of stochatstic programming. Math. Prog., 50, 1997–226.
- Römisch, W., Schultz, R., (1991). Stability analysis for stochatic programms. Annals of OR, 30, 241–266.
- Uryasev, S. 1995. Derivatives of probability functions and some applications. *Annals of Operations Research*, 56(1): 287–311.
- Wendt, M, Li, P., Wozny, G. 2002. Nonlinear chance-constrained pro

Thank you !